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Large time behaviour for a class of turbulence models—stochastic Burgers equations

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Abstract. The aim of this paper is to investigate large time behaviour, i.e. stability and growth bounds, of the solutions for a class of stochastic Burgers equations. The analysis is based on some robustness analysis involved with an infinite-dimensional stochastic evolution equation. Various sufficient conditions for a stochastic Burgers equation are obtained to ensure its asymptotic properties. Lastly, several examples are given to illustrate our theory.

0. Introduction

An important role in fluid dynamics is played by the following *Burgers* equation, see Burgers [4],

$$\partial_t u(t,x) = \nu\partial^2_{xx}u(t,x) + u(t,x)\partial_x u(t,x) \tag{0.1}$$

where $u(t,x)$ is the velocity field and ν is the viscosity. As Burgers emphasized in the introduction of his book [4], this equation represents an extremely simplified model describing the interaction of dissipative and nonlinear inertial terms in the motion of the fluid. A clear discussion on the physical problems connected with Burgers equation can be found in [4]. On the other hand, in some sense it is known, however, that the equation is not a good model for turbulence. It does not display any chaotic phenomena; even when a force is added to the right-hand side and all solutions converge to a unique stationary solution as time goes to infinity. The situation, however, is quite different when the force is random. In particular, a random perturbation may help to select interesting invariant measures. Translational invariance is preserved when (0.1) is perturbed by additive stochastic processes stationary in space and time. Several authors have indeed suggested using the stochastic Burgers equation as a simple model for turbulence: Chambers *et al* [5], Choi *et al* [6] and Dah-Teng Jeng [11].

Roughly speaking, in this paper we hopefully consider the following stochastic evolution equation for $\nu > 0$:

$$\begin{aligned}
&\mathrm{d}u(t,x) = \left(\nu\frac{\partial^2 u(t,x)}{\partial^2 x} + \frac{1}{2}\frac{\partial}{\partial x}u^2(t,x)\right)\mathrm{d}t + g(t,u(t,x))\,\mathrm{d}B_t(x) \qquad t>0,\ x\in(0,1)\\
&u(t,0) = u(t,1) = 0 \qquad t>0\\
&u(0,x) = u_0(x) \qquad x\in[0,1]
\end{aligned} \tag{0.2}$$

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where $u_0(x)$ is a certain given initial function and B_t denotes the Gaussian process defined over a certain probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, with continuous correlation function

$$E(B_t(x), B_{t'}(x')) = t \wedge t' q(x, x') \tag{0.3}$$

where $a \wedge b = \min\{a, b\}$.

The existence, uniqueness and regularity of a sample solution of (0.2) were investigated by Bertini *et al* [2], Brzeźniak *et al* [3] and Da Prato and Gatarek [9]. In the meantime, the asymptotic behaviour of the infinite-dimensional stochastic evolution equation was considered by many authors. Concerning the stability of the stochastic evolution equation, we should notice the fact that lots of authors mainly pay attention to the pth moment stability. In particular we mention Curtain [8], Haussmann [13], Ichikawa [14] and Liu [16] among others. On the other hand, under a number of practical circumstances, we are more concerned with the almost certain stability for a stochastic system. We should also mention Mao's work [17] on the almost certain stability of the n-dimensional stochastic differential equation with respect to a semimartingale.

In this paper, we shall develop a Liapunov functional approach for almost certain stability analysis, and growth bound criteria pertaining to the stochastic Burgers equation (0.2). For simplicity, we shall concern ourselves with the one-dimensional stochastic Burgers equation although it is possible to extend most results to the multi-dimensional equation. Specifically, section 1 contains some mathematical preliminaries for our purposes, such as an infinite-dimensional stochastic integral with respect to a Q-Wiener process and the precise definition of the solution of the stochastic Burgers equation. Section 2 contains the main results of the paper. Based on a basic robustness analysis, the criteria for almost certain asymptotic stability of the stochastic Burgers equation are obtained in theorem 2.1 and theorem 2.2. The growth rate estimates of unbounded solutions to the stochastic equation are also established in theorem 2.3 when stability may be invalid. Finally, section 3 is totally devoted to considering several examples which illustrate how to apply our theory to practical stochastic Burgers equations.

1. Preliminaries

The purpose of this section is to introduce the Hilbert space techniques used to deal with our stochastic Burgers equation. We use the idea from [10]. Let H be the closure of the set $\{u \in C_0^\infty([0,1], R)\colon u(0) = u(1) = 0\}$ in the L^2 norm $|u| = (u, u)^{\frac{1}{2}}$,

$$(u, v) = \int_0^1 u(x)v(x)\,\mathrm{d}x. \tag{1.1}$$

V is the closure of the set $\{u \in C_0^\infty([0,1], R)\colon u(0) = u(1) = 0\}$ in the norm $|u| + \|u\|$, where $\|u\| = ((u, u))^{\frac{1}{2}}$,

$$((u, v)) = \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right). \tag{1.2}$$

Both H and V are Hilbert spaces with their scalar products $(\cdot,\cdot)$, $(\cdot,\cdot)+((\cdot,\cdot))$, respectively.

We denote the self-adjoint extension of the operator $-\Delta$ in H by A and the orthonormal basis of its eigenfunctions with the corresponding eigenvalues $\lambda_k \uparrow +\infty$ by $\{e_k\}$.

We denote the space dual to V by V', with the duality extending the scalar product in H. In general, we define the spaces H^r for $r > 0$ by

$$H^r = \left\{u \in H : \sum_{k=1}^\infty \lambda_k^r u_k^2 < \infty\right\} \tag{1.3}$$

where $u_k = (u, e_k)$ (so $H = H^0$, $V = H^1$). The elements of the dual spaces H^{-r} ($V' = H^{-1}$) are characterized by

$$|u|_{H^{-r}} := \sum_{k=1}^{\infty} \lambda_k^{-r} u_k^2 < \infty \tag{1.4}$$

where $u_k = \langle e_k, u\rangle$, and $\langle\cdot,\cdot\rangle$ denotes the relation of duality between the spaces V and V', so that $u = \sum_{k=1}^{\infty} u_k e_k$, the limit being taken in H^{-r}.

The operator A can be extended to a continuous linear operator, still denoted by A, from V into V' by $\langle v, Au\rangle = ((v, u))$ for $u, v \in V$. We also define

$$b(u, v, z) = \int_0^1 u(x) \frac{\partial v}{\partial x}(x) z(x)\, \mathrm{d}x = ((\langle u, \nabla\rangle v, z)$$

whenever the integrals make sense. Note that for $u, v, z \in V$ we have $b(u, v, z) = -b(u, z, v)$, hence $b(u, v, v) = 0$. We also have the following well known inequality for $b(u, v, z)$ and we list them here for reference:

$$|b(u, v, z)| \leqslant c\|u\|\|v\|\|z\| \tag{1.5}$$

$$|b(u, v, z)| \leqslant c|u|\|v\||Az| \tag{1.6}$$

$$|b(u, v, z)| \leqslant c\|u\||v||Az| \tag{1.7}$$

for suitable u, v, z and constant c. The inequality (1.5) allows us to define a V'-valued bilinear form $B(u, v)$ by $\langle z, B(u, v)\rangle = b(u, v, z)$.

Let us now introduce the following notations for the path spaces:

$$\begin{aligned} L_T &= L^\infty(0, T; H) \\ \mathcal{L}_T &= L^\infty(0, T; H) \cap L^2(0, T; V) \cap C(0, T; H) \\ \mathcal{L} &= \bigcap_{T<\infty} \mathcal{L}_T. \end{aligned} \tag{1.8}$$

Definition 1.1. Let K be a separable Hilbert space. A stochastic process W_t, $t \geqslant 0$, in Hilbert space K is a Q-Wiener process defined on $(\Omega, \mathcal{F}, P)$ if:

(a) W_t is a square integrable process and $EW_t = 0$ for all $t \geqslant 0$;
(b) $\mathrm{Cov}[W_t - W_s] = (t - s)Q$, $Q \in \mathcal{L}(K)$ is a non-negative nuclear operator;
(c) W_t has continuous sample paths;
(d) W_t has independent increments;

where $\mathcal{L}(K) = \mathcal{L}(K, K)$ is the family of all bounded linear operators from K into itself, equipped with the usual operator norm topology. The operator Q is the incremental covariance operator of the Wiener process W_t.

Let $\mathcal{F}_t[W_\cdot]$ be the σ-field generated by W_s, $0 \leqslant s \leqslant t$; then W_t is a martingale relative to $\mathcal{F}_t[W_\cdot]$. We have the following representation of a Wiener process.

Proposition 1.1. Let W_t be a Wiener process in K with incremental covariance operator Q, then

$$W_t = \sum_{i=1}^{\infty} \beta_i(t) e_i \tag{1.9}$$

where $\{e_i\}$ is an orthonormal set of eigenvectors of Q, $\beta_i(t)$ are mutually independent real Wiener processes with incremental covariance $\lambda_i > 0$, $Qe_i = \lambda_i e_i$ and $\mathrm{tr}\, Q = \sum_{i=1}^{\infty} \lambda_i$.

The stochastic integral $\int_0^t g(s)\,\mathrm{d}W_s$ is defined as follows. First we introduce the space of integrands. For any Hilbert space U, we denote by $\mathcal{U}(U)$ the space of all stochastic processes

$$g(t,\omega) : [0,T] \times \Omega \to \mathcal{L}(K,U)$$

such that

$$E\left(\int_0^T \|g(t)\|^2_{\mathcal{L}(K,U)}\,\mathrm{d}t\right) < \infty$$

where $\mathcal{L}(K,U)$ is the space consisting of all bounded linear operators from K into U, equipped with the usual operator norm topology, and for all $k \in K$, $g(t)k$ is a U-valued stochastic process measurable with respect to the filtration $\mathcal{F}_t$.

The stochastic integral $\int_0^t g(s)\,\mathrm{d}W_s \in U$ is defined for all $g \in \mathcal{U}(U)$ by

$$\int_0^t g(s)\,\mathrm{d}W_s = L^2 - \lim_{n\to\infty} \sum_{i=1}^n \int_0^t g(s)e_i\,\mathrm{d}\beta_i(s). \tag{1.10}$$

Roughly speaking, in this paper we shall actually study a class of much more extended stochastic evolution equations as follows for $t \geqslant t_0 \geqslant 0$:

$$\mathrm{d}Y_t(\omega) = [-\nu A Y_t(\omega) + f(t, Y_t(\omega))]\,\mathrm{d}t + g(t, Y_t(\omega))\,\mathrm{d}W_t. \tag{1.11}$$

In particular, we give the following.

Definition 1.2. Let $f(t,y) : R^+ \times V \to V'$, $g(t,y) : R^+ \times V \to \mathcal{L}(K,H)$ be two Borel measurable functions such that for all $t \in R^+$ and $y \in V$, $\langle y, f(t,y)\rangle = 0$, and $g(t,Y_t) \in \mathcal{U}(H)$. A Hilbert space-valued stochastic process Y_t with almost sure paths in $\mathcal{L}$ is *a solution* of the stochastic Burgers equation (1.11) if, for $t \geqslant t_0 \geqslant 0$,

$$Y_t(\omega) = Y_{t_0}(\omega) + \int_{t_0}^t [-\nu A Y_s(\omega) + f(s, Y_s(\omega))]\,\mathrm{d}s + \int_{t_0}^t g(s, Y_s(\omega))\,\mathrm{d}W_s \tag{1.12}$$

holds as an identity in V' (the first integral is understood in the sense of Bochner).

As a consequence, we are now in the position to formulate (0.2) as a stochastic evolution equation in the Hilbert space V':

$$\mathrm{d}Y_t = [-\nu A Y_t + f(t,Y_t)]\,\mathrm{d}t + g(t,Y_t)\,\mathrm{d}W_t \tag{1.13}$$

where $f(t,Y_t) = \frac{1}{2}(\partial/\partial x)Y_t^2(x)$ and $g(t,y)\colon R^+ \times V \to H$ is a Borel measurable function with $g(t,Y_t) \in \mathcal{U}(H)$. W_t is an H-valued Wiener process with the covariance operator Q such that for all $v(x) \in H$

$$(Qv)(x) = \int_0^1 q(x,y)v(y)\,\mathrm{d}y.$$

2. The main results

In this section, we shall try to obtain our main results of the stochastic Burgers equation. Owing to the fact that we are restricting ourselves to stability analysis, we assume that the equation has a unique global solution which is denoted by $Y_t(Y_{t_0})$, or $Y(t, Y_{t_0})$.

Theorem 2.1. Let $Y_t(\omega)$ be a global soluton of (1.12). Assume there exist a real function $\psi(t) \geqslant 0$ and a non-negative constant $\lambda > 0$ such that:

$$\|g(t, y)\|^2_{\mathcal{L}(K,H)} \leqslant \psi(t)\,\mathrm{e}^{-2\lambda t} \qquad y \in V, t \in R^+ \tag{2.1}$$

where $\psi(t)$ satisfies for any $\delta > 0$

$$\lim_{t\to\infty} \frac{\psi(t)}{\mathrm{e}^{\delta t}} = 0.$$

Then we have that the solution of equation (1.12) is exponentially stable almost certainly. Moreover,

$$\limsup_{t\to\infty} \frac{\log|Y_t|}{t} \leqslant -\nu\lambda_0 \qquad \text{a.s.} \tag{2.2}$$

where $\lambda_0 = \inf_{y\in V}(|\nabla y(x)|^2/|y(x)|^2) > 0$.

Proof. For simplicity, we suppose $Y_0 = 0$. For any $\delta > 0$ small enough, we define a continuous functional on the space V', $V(v,t)(\cdot) = \mathrm{e}^{2(\lambda-\delta)t}\langle v, \cdot\rangle^2$, where $t \geqslant 0$, $v \in V$ and $\langle\cdot,\cdot\rangle$ denotes the canonical pairing between V and V'. Using Itô's formula, we can derive that

$$\begin{aligned} V(e_i, t)(Y_t) \leqslant \int_0^t &\{2(\lambda-\delta)\,\mathrm{e}^{2(\lambda-\delta)s}\langle e_i, Y_s\rangle^2 + 2\,\mathrm{e}^{2(\lambda-\delta)s}\langle e_i, Y_s\rangle\langle e_i, -\nu A Y_s + f(s, Y_s)\rangle \\ &+\mathrm{e}^{2(\lambda-\delta)s}\|g(s, Y_s)\|^2_{\mathcal{L}(K,H)}\,\mathrm{tr}[(e_i\otimes e_i)\cdot Q]\}\,\mathrm{d}s \\ &+2\int_0^t \mathrm{e}^{2(\lambda-\delta)s}\langle e_i, Y_s\rangle(e_i, g(s, Y_s)\,\mathrm{d}W_s) \end{aligned} \tag{2.3}$$

where $\{e_i\} \in V$ is the orthonormal basis of the eigenfunctions of the operator A with corresponding eigenvalues $\lambda_i \uparrow +\infty$. Taking account of $Y_t \in \mathcal{L}$ a.s., we derive almost certainly

$$\begin{aligned} V(Y_t, t)(Y_t) \leqslant \int_0^t &\{2(\lambda-\delta)\,\mathrm{e}^{2(\lambda-\delta)s}|Y_s|^2 + 2\,\mathrm{e}^{2(\lambda-\delta)s}\langle Y_s, -\nu A Y_s + f(s, Y_s(\omega))\rangle \\ &+\mathrm{e}^{2(\lambda-\delta)s}\|g(s, Y_s)\|^2_{\mathcal{L}(K,H)}\,\mathrm{tr}\,Q\}\mathrm{d}s + 2\int_0^t \mathrm{e}^{2(\lambda-\delta)s}(Y_s, g(s, Y_s)\,\mathrm{d}W_s). \end{aligned} \tag{2.4}$$

Owing to the exponential martingale inequality, we have

$$\begin{aligned} P\Bigg\{\omega\colon \sup_{0\leqslant t\leqslant w}\Bigg(&\int_0^t \mathrm{e}^{2(\lambda-\delta)s}(Y_s, g(s, Y_s)\,\mathrm{d}W_s) \\ &-\int_0^t \frac{u}{2}\,\mathrm{e}^{4(\lambda-\delta)s}\,\mathrm{tr}[g(s, Y_s)Qg(s, Y_s)^*]\cdot(Y_s, Y_s)\,\mathrm{d}s\Bigg) > v\Bigg\} \leqslant \mathrm{e}^{-uv} \end{aligned}$$

for any positive constants u, v and w. In particular, choosing

$$u = 2 \qquad v = \log k \qquad w = \frac{k}{2^n} \qquad k = 1, 2, \ldots, n = 1, 2, \ldots$$

we then apply the well known Borel–Cantelli lemma to obtain the fact that there exists an integer $k_0(n, \omega)$ for almost all $\omega \in \Omega$ such that

$$\int_0^t (\mathrm{e}^{2(\lambda-\delta)s}Y_s, g(s, Y_s)\,\mathrm{d}W_s) \leqslant \log k + \mathrm{tr}\,Q\int_0^t \mathrm{e}^{4(\lambda-\delta)s}\|g(s, Y_s)\|^2_{\mathcal{L}(K,H)}(Y_s, Y_s)\,\mathrm{d}s$$

for all $0 \leqslant t \leqslant k/2^n$, $k \geqslant k_0(n,\omega)$. Substituting this into (2.4) and using hypotheses of the theorem, we see that almost certainly

$$\begin{aligned}
e^{2(\lambda-\delta)t}(Y_t, Y_t) \leqslant & \int_0^t e^{2(\lambda-\delta)s}\{2(\lambda-\delta)(Y_s, Y_s) \\
& +2\langle Y_s, -\nu A Y_s\rangle + \|g(s, Y_s)\|^2_{\mathcal{L}(K,H)} \operatorname{tr} Q\}\,\mathrm{d}s + 2\log k \\
& +2 \operatorname{tr} Q \int_0^t e^{4(\lambda-\delta)s}\|g(s, Y_s)\|^2_{\mathcal{L}(K,H)}(Y_s, Y_s)\,\mathrm{d}s \\
\leqslant & \; 2\log k + \int_0^t (2(\lambda-\delta) - 2\nu\lambda_0 + 2\operatorname{tr} Q \cdot \psi(s)\cdot e^{-2\delta s})\, e^{2(\lambda-\delta)s}(Y_s, Y_s)\,\mathrm{d}s \\
& + \operatorname{tr} Q \int_0^t \psi(s)\cdot e^{-2\delta s}\,\mathrm{d}s.
\end{aligned}$$

So by Gronwall's inequality, we derive that almost certainly

$$\begin{aligned}
e^{2(\lambda-\delta)t}(Y_t, Y_t) \leqslant & \left(2\log k + \operatorname{tr} Q \int_0^t \psi(s)\cdot e^{-2\delta s}\,\mathrm{d}s\right) \\
& \times \exp\left\{\int_0^t (2(\lambda-\delta) - 2\nu\lambda_0 + 2\operatorname{tr} Q\cdot\psi(s)\cdot e^{-2\delta s})\,\mathrm{d}s\right\} \qquad (2.5)
\end{aligned}$$

for $0 \leqslant t \leqslant k/2^n$, $k \geqslant k_0(n,\omega)$.

On the other hand, for arbitrary $\epsilon > 0$ there exists a positive integer N and a random integer $k_1 = k_1(N,\omega)$ such that if $k/2^N \leqslant t \leqslant (k+1)/2^N$, $k \geqslant k_1(N,\omega) \vee k_0(N,\omega)$ we have

$$\left|\frac{k}{2^N} - t\right| \leqslant \epsilon$$

and, furthermore, this implies that there exists a positive constant $M > 0$ such that

$$\begin{aligned}
\log(e^{2(\lambda-\delta)t}(Y_t, Y_t)) \leqslant & \log[2\log k + \operatorname{tr} Q\cdot t\cdot M] + 2(\lambda-\delta)t - 2\nu\lambda_0 t \\
& +2\operatorname{tr} Q \cdot \int_0^t \psi(s)\, e^{-2\delta s}\,\mathrm{d}s
\end{aligned}$$

where $k/2^N \leqslant t \leqslant (k+1)/2^N$, $k \geqslant k_1(N,\omega)\vee k_0(N,\omega)$. Therefore

$$\limsup_{t\to\infty} \frac{\log(e^{2(\lambda-\delta)t}(Y_t, Y_t))}{t} \leqslant 2(\lambda-\delta) - 2\nu\lambda_0 + O(\epsilon).$$

Letting $\epsilon \to 0$, $\delta \to 0$ gives

$$\limsup_{t\to\infty} \frac{\log(e^{2\lambda t}(Y_t, Y_t))}{t} \leqslant 2\lambda - 2\nu\lambda_0.$$

Finally, we have

$$\limsup_{t\to\infty} \frac{\log|Y_t|}{t} = \limsup_{t\to\infty} \frac{1}{2}\frac{\log(e^{2\lambda t}(Y_t, Y_t))}{t} - \lambda \leqslant -\nu\lambda_0.$$

Now the proof is complete. □

Theorem 2.2. Assume there exists a positive constant $\alpha > 0$ such that:

(1) $\|g(t, y)\|^2_{\mathcal{L}(K,H)} \leqslant 2\alpha|y|^2$, $(t, y) \in R^+ \times V$;

(2) $\nu\lambda_0 - \alpha \operatorname{tr} Q > 0$; where $\lambda_0 = \inf_{y\in V}(|\nabla y(x)|^2/|y(x)|^2) > 0$. Then there exists a pair of positive constants M and γ such that

$$E|Y_t(t_0)|^2 \leqslant M\,\mathrm{e}^{-\gamma(t-t_0)}E|Y_{t_0}|^2 \tag{2.6}$$

for all $t_0 \geqslant 0$. In this case, we say equation (1.12) is the second momently stable. Moreover, we have

$$\limsup_{t\to\infty} \frac{\log|Y_t|}{t} \leqslant -\frac{\nu\lambda_0 - \alpha \operatorname{tr} Q}{2} \qquad \text{a.s.} \tag{2.7}$$

In other words, equation (1.12) is also exponentially stable almost certainly.

Proof. First of all, by condition (2) we can find $\gamma > 0$ such that

$$2\nu\lambda_0 - 2\alpha \operatorname{tr} Q - \gamma > 0.$$

We now claim that there exists $C > 0$ such that

$$\int_{t_0}^{\infty} \mathrm{e}^{\gamma s}E|Y_s(t_0)|^2\,\mathrm{d}s \leqslant C\cdot \mathrm{e}^{\gamma t_0}E|Y_{t_0}|^2.$$

Indeed, Itô's formula and condition (1) imply that for any $\lambda > 0$

$$\mathrm{e}^{\lambda t}E|Y_t|^2 \leqslant \mathrm{e}^{\lambda t_0}E|Y_{t_0}|^2 + \int_{t_0}^{t}(\lambda - 2\nu\lambda_0)\,\mathrm{e}^{\lambda s}E|Y_s|^2\,\mathrm{d}s + \operatorname{tr} Q\cdot E\int_{t_0}^{t} \mathrm{e}^{\lambda s}\|g(s, Y_s)\|^2_{\mathcal{L}(K,H)}\,\mathrm{d}s.$$

Hence, by virtue of condition (1), we deduce

$$E|Y_t|^2 \leqslant \mathrm{e}^{-\lambda(t-t_0)}E|Y_{t_0}|^2 + \int_{t_0}^{t}(\lambda - 2\nu\lambda_0 + 2\alpha\cdot \operatorname{tr} Q)\,\mathrm{e}^{-\lambda(t-s)}E|Y_s|^2\,\mathrm{d}s.$$

Thus, for any $T > t_0$ and $\gamma > 0$ satisfying $\gamma \in (0, (2\nu\lambda_0 - 2\alpha \operatorname{tr} Q)\wedge\lambda)$, we have

$$\begin{aligned}\int_{t_0}^{T} \mathrm{e}^{\gamma t}E|Y_t|^2\,\mathrm{d}t &\leqslant \int_{t_0}^{T} \mathrm{e}^{\gamma t-\lambda(t-t_0)}E|Y_{t_0}|^2\,\mathrm{d}t + (\lambda - 2\nu\lambda_0 + 2\alpha\cdot \operatorname{tr} Q)\\ &\quad\times\int_{t_0}^{T} \mathrm{e}^{\gamma t}\int_{t_0}^{t} \mathrm{e}^{-\lambda(t-s)}E|Y_s(t_0)|^2\,\mathrm{d}s\,\mathrm{d}t\\ &\leqslant \frac{1}{\lambda-\gamma}\,\mathrm{e}^{\gamma t_0}E|Y_{t_0}|^2 + \frac{\lambda - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q}{\lambda-\gamma}\int_{t_0}^{T} \mathrm{e}^{\gamma s}E|Y_s(t_0)|^2\,\mathrm{d}s\end{aligned}$$

which immediately implies that

$$\int_{t_0}^{T} \mathrm{e}^{\gamma t}E|Y_t|^2\,\mathrm{d}t \leqslant \frac{(\,\mathrm{e}^{\gamma t_0}E|Y_{t_0}|^2/\lambda-\gamma)}{1-(\lambda - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q/\lambda-\gamma)} = \frac{\mathrm{e}^{\gamma t_0}E|Y_{t_0}|^2}{2\nu\lambda_0 - \gamma - 2\alpha \operatorname{tr} Q}.$$

In a similar way, we can derive that for $\tilde{\gamma} > 2\nu\lambda_0 - 2\alpha \operatorname{tr} Q$

$$\mathrm{e}^{\tilde{\gamma} t}E|Y_t|^2 \leqslant \mathrm{e}^{\tilde{\gamma} t_0}E|Y_{t_0}|^2 + \int_{t_0}^{t}(\tilde{\gamma} - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q)\,\mathrm{e}^{\tilde{\gamma} s}E|Y_s|^2\,\mathrm{d}s$$

which, combined with the preceding results, immediately implies that there exists $C(\gamma) > 0$ such that

$$\begin{aligned}\mathrm{e}^{\tilde{\gamma} t}E|Y_t|^2 &\leqslant \mathrm{e}^{\tilde{\gamma} t_0}E|Y_{t_0}|^2 + (\tilde{\gamma} - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q)\int_{t_0}^{t} \mathrm{e}^{(\tilde{\gamma}-\gamma)s}\cdot\,\mathrm{e}^{\gamma s}\cdot E|Y_s|^2\,\mathrm{d}s\\ &\leqslant \mathrm{e}^{\tilde{\gamma} t_0}E|Y_{t_0}|^2 + (\tilde{\gamma} - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q)\cdot C(\gamma)\cdot \mathrm{e}^{\gamma t_0+(\tilde{\gamma}-\gamma)t}\cdot E|Y_{t_0}|^2\end{aligned}$$

where $C(\gamma) = 1/(2\nu\lambda_0 - \gamma - 2\alpha \operatorname{tr} Q)$, that is,

$$E|Y_t|^2 \leqslant E|Y_{t_0}|^2(\mathrm{e}^{-\tilde{\gamma}(t-t_0)} + C(\gamma)\cdot(\tilde{\gamma} - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q)\,\mathrm{e}^{-\gamma(t-t_0)}) \leqslant M\,\mathrm{e}^{-\gamma(t-t_0)}E|Y_{t_0}|^2$$

where $M = 1 + C(\gamma)\cdot(\tilde{\gamma} - 2\nu\lambda_0 + 2\alpha \operatorname{tr} Q) > 0$.

Finally, we show that (2.6) implies (2.7), i.e. (2.6) implies the almost certain exponential stability. To this end, we divide our proof into two steps.

Step 1. We claim that there exists a positive constant $K_0 < \infty$ such that

$$E\left(\sup_{t_0\leqslant t<\infty}|Y_t|^2\right) \leqslant K_0 E|Y_{t_0}|^2.$$

Indeed, by virtue of Itô's formula,

$$\begin{aligned}|Y_t|^2 &= |Y_{t_0}|^2 - 2\nu\int_{t_0}^t \langle Y_s, AY_s\rangle\,\mathrm{d}s + 2\int_{t_0}^t (Y_s, g(s, X_s)\,\mathrm{d}W_s)\\ &\quad + \int_{t_0}^t \operatorname{tr}(g(s, Y_s)Qg(s, Y_s)^*)\,\mathrm{d}s\\ &\leqslant |Y_{t_0}|^2 - (2\nu\lambda_0 - 2\alpha \operatorname{tr} Q)\int_{t_0}^t |Y_s|^2\,\mathrm{d}s + 2\int_{t_0}^t (Y_s, g(s, X_s)\,\mathrm{d}W_s).\end{aligned}$$

Hence, for arbitrary $T > t_0$

$$E\sup_{t_0\leqslant t<T}|Y_t|^2 \leqslant E|Y_{t_0}|^2 + 2\alpha \operatorname{tr} Q\int_{t_0}^T E|Y_s|^2\,\mathrm{d}s + 2E\sup_{t_0\leqslant t<T}\left|\int_{t_0}^t (Y_s, g(s, X_s)\,\mathrm{d}W_s)\right|. \quad (2.8)$$

But by virtue of the Burkholder–Davis–Gundy inequality, we easily obtain

$$\begin{aligned}2E&\sup_{t_0\leqslant t<T}\left|\int_{t_0}^t (Y_s, g(s, X_s)\,\mathrm{d}W_s)\right|\\ &\leqslant 6E\left\{\int_{t_0}^T |Y_s|^2 \operatorname{tr}(g(s, Y_s)Qg(s, Y_s)^*)\,\mathrm{d}s\right\}^{\frac{1}{2}}\\ &\leqslant 3E\left\{2\sup_{t_0\leqslant t\leqslant T}|Y_s|\left[\int_{t_0}^T \operatorname{tr}(g(s, Y_s)Qg(s, Y_s)^*)\,\mathrm{d}s\right]^{\frac{1}{2}}\right\}\\ &\leqslant 3lE\left\{\sup_{t_0\leqslant t\leqslant T}|Y_t|^2\right\} + 12\,l^{-1}\alpha \operatorname{tr} Q\int_{t_0}^T E|Y_s|^2\,\mathrm{d}s.\end{aligned}$$

If we take $l = \frac{1}{6}$ and substitute into (2.8) we obtain, after using (2.6), that there exists $K_0 > 0$ such that

$$E\left(\sup_{t_0\leqslant t<\infty}|Y_t|^2\right) \leqslant K_0 E|Y_{t_0}|^2.$$

Step 2. Following a similar argument, it easily follows that for $t \geqslant N$

$$\begin{aligned}|Y_t|^2 &= |Y_N|^2 - 2\nu\int_N^t \langle Y_s, AY_s\rangle\,\mathrm{d}s + 2\int_N^t (Y_s, g(s, X_s)\,\mathrm{d}W_s)\\ &\quad + \int_N^t \operatorname{tr}(g(s, Y_s)Qg(s, Y_s)^*)\,\mathrm{d}s\\ &\leqslant |Y_N|^2 + 2\alpha \operatorname{tr} Q\int_N^t |Y_s|^2\,\mathrm{d}s + 2\int_N^t (Y_s, g(s, X_s)\,\mathrm{d}W_s).\end{aligned}$$

Hence, let $\epsilon_N > 0$ be arbitrary. Then

$$P\left\{\omega: \sup_{N\leqslant t\leqslant N+1}|Y_t(t_0)| > \epsilon_N\right\} \leqslant P\{|Y_N|^2 \geqslant \epsilon_N^2/3\} + P\left\{\int_N^{N+1}|Y_t|^2\,\mathrm{d}t \geqslant \epsilon_N^2/6\alpha\,\mathrm{tr}\,Q\right\}$$
$$+P\left\{\sup_{N\leqslant t\leqslant N+1}\int_N^t (Y_s, g(s, X_s)\,\mathrm{d}W_s) \geqslant \epsilon_N^2/6\right\}.$$

Now, using the consequences derived above,

$$P\left\{\sup_{N\leqslant t\leqslant N+1}\int_N^t (Y_s, g(s, X_s)\,\mathrm{d}W_s) \geqslant \epsilon_N^2/6\right\}$$
$$\leqslant 36\alpha\,\mathrm{tr}\,Q\epsilon_N^{-2}\left\{E\sup_{N\leqslant t\leqslant N+1}|Y_t(t_0)|\right\}\left\{\int_N^{N+1}E|Y_t|^2\,\mathrm{d}t\right\}^{\frac{1}{2}}$$
$$\leqslant \frac{k_1 E|Y_{t_0}|^2\,\mathrm{e}^{-\gamma N/2}}{\epsilon_N^2}.$$

If $\epsilon_N = k_1\,\mathrm{e}^{-\gamma N\delta/4}$ where $0 < \delta < 1$, the Borel–Cantelli lemma now implies that there are $N'(\omega)$ and $M > 0$ such that if $N > N'(\omega)$, then

$$\sup_{N\leqslant t\leqslant N+1}|Y_t|^2 \leqslant M\,\mathrm{e}^{-\gamma N\delta/2}.$$

Consequently, letting $\delta \to 1$,

$$\frac{1}{t}\log|Y_t(t_0)| \leqslant \frac{1}{(N-1)+t_0}\left(-\frac{1}{4}\gamma N\right)$$

whenever $(N-1)+t_0 \leqslant t \leqslant N+t_0$, $N \geqslant N'(\omega)$ almost certainly. Therefore,

$$\limsup_{t\to\infty}\frac{\log|Y_t|}{t} \leqslant -\frac{\gamma}{4} \qquad \text{a.s.}$$

Furthermore, notice that $\gamma \in (0, 2\nu\lambda_0 - 2\alpha\,\mathrm{tr}\,Q)$ is arbitrary, (2.7) is derived easily by letting γ tend to $2\nu\lambda_0 - 2\alpha\,\mathrm{tr}\,Q$. Now the proof is complete. □

As is well known, under some conditions such as stochastic bounded perturbation, the supreme $Y_t^* = \sup_{0\leqslant s\leqslant t}|Y_s|$ of the solution may tend to infinity almost certainly and therefore it is useful to establish upper bounds for the supremum. To the end, let us consider the following extended stochastic equation,

$$Y_t = y_0 + \int_0^t (-\nu T Y_s + f(s, Y_s))\,\mathrm{d}s + \int_0^t g(s, Y_s)\,\mathrm{d}W_s \tag{2.9}$$

which holds as an identity in V', where T is a linear operator, which is in general unbounded, defined on a dense linear subspace $\mathcal{D}(T) \subset V \subset H$ which has a self-adjoint extension, still simply denoted by T, on V such that T is a continuous linear operator from V into V'. $Y_t \in V$ a.e. and W_t is a K-valued Q-Wiener process with $W_0 = 0$ and $f: R^+ \times V \to V'$, $g : R^+ \times V \to \mathcal{L}(K, V')$ are two continuous, locally bounded mappings with suitable regular hypotheses. Once again we still hopefully assume that equation (2.9) has a unique global solution, defined in the obvious manner similar to definition 1.2, which is denoted by $Y_t \in V$ a.e. Without loss of generality, we might as well assume $Y_0 = 0$ for simplicity.

Assuming $V(y, t)$ is a $C^{2,1}$-positive function defined on $H \times R^+$ such that $V_y'(y, t) \in V$ for all $y \in V$, $t \in R^+$, we define operators L and Q as follows for $y \in V$, $t \in R^+$:

$$LV(y, t) = V_t'(y, t) + \langle V_y'(y, t), -\nu T y + f(y, t)\rangle + \tfrac{1}{2}\,\mathrm{tr}[V_{yy}''(y, t)(g \circ Q^{\frac{1}{2}})(g \circ Q^{\frac{1}{2}})^*]$$

$$(2.10)$$

and

$$QV(y,t) = \mathrm{tr}[V_y' \otimes V_y'(y,t)(g \circ Q^{\frac{1}{2}})(g \circ Q^{\frac{1}{2}})^*]. \tag{2.11}$$

Lemma 2.1. Let $V(y,t) \in C^{2,1}(H \times R^+; R^+)$ such that $V_y'(y,t) \in V$ for $y \in V$, and $\psi_1(t)$ and $\psi_2(t)$ be two non-negative continuous functions. Let $\lambda(t) \uparrow +\infty$ be a positive, increasing function. Assume that for all $y \in V$ and $t \geqslant 0$ there exist positive constants $p > 0$, $M > 0$, $\theta > 0$, $\nu \geqslant 0$, $\mu \geqslant 0$ and positive function $\xi(t) \downarrow 0$ such that

(1) $|y|^p \cdot \xi(t)^{-1} = V(y,t)$, $(y,t) \in V \times R^+$;

(2) $LV(y,t) + \xi(t)QV(y,t) \leqslant \psi_1(t) + \psi_2(t)V(y,t)$, $(y,t) \in V \times R^+$;

(3)

$$\sum_{k=1}^{\infty} \frac{1}{\lambda(k)^2} < \infty \qquad \limsup_{k\to\infty} \frac{\log\log\lambda(k)}{\log\log\lambda(k-1)} \leqslant \theta < \infty$$

(4)

$$\limsup_{t\to\infty} \frac{\log(\xi(t)\int_0^t \psi_1(s)\,\mathrm{d}s)}{\log\log\lambda(t)} \leqslant \nu \qquad \limsup_{t\to\infty} \frac{\xi(t)\int_0^t \psi_2(s)\xi(s)^{-1}\,\mathrm{d}s}{\log\log\lambda(t)} \leqslant \mu$$

$$\limsup_{k\to\infty} \frac{\xi(k-1)}{\xi(k)} \leqslant M < \infty \qquad k = 1, 2, \ldots.$$

Then there exists a constant random variable $C(\omega)$ such that the solution of equation (2.9) satisfies

$$Y_t^* \leqslant C(\omega) \cdot (\log\lambda(t))^{(\nu\vee\theta+\mu)/p} \qquad \text{a.s.}$$

where $Y_t^* = \sup_{0\leqslant s\leqslant t}|Y_s|$.

Proof. By Itô's formula and the definition of L and Q, we can derive

$$V(Y_t,t) = V(Y_0,0) + \int_0^t LV(Y_s,s)\,\mathrm{d}s + \int_0^t \langle V_y'(Y_s,s), g(Y_s,s)\,\mathrm{d}W_s\rangle. \tag{2.12}$$

Due to the exponential martingale inequality, we have

$$P\left\{\omega\colon \sup_{0\leqslant t\leqslant w}\left[\int_0^t \langle V_y'(Y_s,s), g(Y_s,s)\,\mathrm{d}W_s\rangle - \int_0^t \frac{u}{2}QV(Y_s,s)\,\mathrm{d}s\right] > v\right\} \leqslant \mathrm{e}^{-uv}$$

for any positive constants u, v and w. In particular, choosing

$$u = 2\xi(k) \qquad v = \xi(k)^{-1}\log\lambda(k) \qquad w = k \qquad k = 1, 2, \ldots$$

we then apply the well known Borel–Cantelli lemma to obtain the fact that there exists an integer $k_0(\omega)$ for almost all $\omega \in \Omega$ such that

$$\int_0^t \langle V_y'(Y_s,s), g(Y_s,s)\,\mathrm{d}W_s\rangle \leqslant \xi(k)^{-1}\log\lambda(k) + \xi(k)\int_0^t QV(Y_s,s)\,\mathrm{d}s$$

for all $0 \leqslant t \leqslant k$, $k \geqslant k_0$. Substituting this into (2.12) and using hypotheses of the lemma 2.1, we see that almost certainly

$$\begin{aligned}
V(Y_t,t) &\leqslant \xi(k)^{-1}\log\lambda(k) + \int_0^t LV(Y_s,s)\,\mathrm{d}s + \int_0^t \xi(k)QV(Y_s,s)\,\mathrm{d}s \\
&\leqslant \xi(k)^{-1}\log\lambda(k) + \int_0^t LV(Y_s,s)\,\mathrm{d}s + \int_0^t \xi(s)QV(Y_s,s)\,\mathrm{d}s \\
&\leqslant \xi(k)^{-1}\log\lambda(k) + \int_0^t (\psi_1(s) + \psi_2(s)V(Y_s,s))\,\mathrm{d}s
\end{aligned}$$

that is,

$$|Y_t|^p \leqslant \xi(t)\xi(k)^{-1}\log\lambda(k) + \xi(t)\int_0^t (\psi_1(s) + \psi_2(s)\cdot\xi(s)^{-1}\cdot|Y_s|^p)\,\mathrm{d}s.$$

By hypotheses of the lemma 2.1 and using Gronwall's inequality, we derive that almost certainly

$$|Y_t|^p \leqslant \left[\xi(t)\xi(k)^{-1}\log\lambda(k) + \xi(t)\int_0^t \psi_1(s)\,\mathrm{d}s\right]\cdot\exp\left(\xi(t)\int_0^t \psi_2(s)\xi(s)^{-1}\,\mathrm{d}s\right)$$

for $0 \leqslant t \leqslant k$, $k \geqslant k_0(\omega)$.

By conditions (3) and (4), for any $\epsilon > 0$ there exists a random integer $k_1 = k_1(\omega)$ such that if $k-1 \leqslant t \leqslant k$, $k \geqslant k_1 \vee k_0$, we have

$$\begin{aligned}\log|Y_t|^p &\leqslant \log[(M+\epsilon)\log\lambda(k) + (\log\lambda(t))^{(\nu+\epsilon)}] + \xi(t)\int_0^t \psi_2(s)\xi(s)^{-1}\,\mathrm{d}s\\ &\leqslant \log[(M+\epsilon)(\log\lambda(t))^{(\theta+\epsilon)} + (\log\lambda(t))^{(\nu+\epsilon)}] + (\mu+\epsilon)\log\log\lambda(t)\end{aligned}$$

which implies immediately that

$$\limsup_{t\to\infty}\frac{\log|Y_t|^p}{\log\log\lambda(t)} \leqslant (\theta+\epsilon)\vee(\nu+\epsilon)+\mu+\epsilon \qquad \text{a.s.}$$

Letting $\epsilon \to 0$ and using lemma 6.3 of [18] gives

$$\limsup_{t\to\infty}\frac{\log(Y_t^*)^p}{\log\log\lambda(t)} \leqslant \theta\vee\nu+\mu \qquad \text{a.s.}$$

Finally, we have the fact that there exists a random variable $C(\omega)$ such that

$$Y_t^* \leqslant C(\omega)\cdot(\log\lambda(t))^{(\theta\vee\nu+\mu)/p} \qquad \text{a.s.}$$

The proof is complete. □

Next, we shall apply the preceding lemma 2.1 to our stochastic Burgers equation (1.12). In particular, we have the following consequence.

Theorem 2.3. Suppose there exists a positive $M > 0$ such that

$$\|g(t,y)\|_{\mathcal{L}(K,H)} \leqslant M \qquad t\in R^+ \qquad y\in V$$

then the solution of equation (1.12) satisfies the fact that there exists a random variable $C(\omega)$ such that

$$Y_t^* \leqslant C(\omega)\cdot\sqrt{\log t} \qquad \text{a.s.}$$

where $Y_t^* = \sup_{0\leqslant s\leqslant t}|Y_s|$.

Proof. Let $V(t,v)(\cdot) = \mathrm{e}^{\lambda t}\langle v,\cdot\rangle^2$, where $t\geqslant 0$, $v\in V$ and λ is some positive constant fixed afterwards. It is easy to obtain the fact that for a solution $Y_t\in\mathcal{L}$ a.s.

$$\begin{aligned}&LV(t,Y_t)(Y_t) + \mathrm{e}^{-\lambda t}QV(t,Y_t)(Y_t)\\ &\quad= \lambda\,\mathrm{e}^{\lambda t}\langle Y_t,Y_t\rangle + \mathrm{e}^{\lambda t}\langle Y_t, -2\nu AY_t + f(t,Y_t(\omega))\rangle + M^2\,\mathrm{tr}\,Q\,\mathrm{e}^{\lambda t}\\ &\qquad +2M^2\,\mathrm{tr}\,Q\,\mathrm{e}^{\lambda t}|Y_t|^2\\ &\quad\leqslant \lambda\,\mathrm{e}^{\lambda t}\langle Y_t,Y_t\rangle - 2\nu\langle\nabla Y_t,\nabla Y_t\rangle\,\mathrm{e}^{\lambda t} + 2M^2\,\mathrm{tr}\,Q|Y_t|^2\,\mathrm{e}^{\lambda t}|Y_t|^2 + M^2\,\mathrm{tr}\,Q\,\mathrm{e}^{\lambda t}\\ &\quad\leqslant (\lambda - 2\nu\lambda_0 + 2\,\mathrm{tr}\,Q\cdot M^2)V(t,Y_t)(Y_t) + M^2\cdot\mathrm{tr}\,Q\cdot\,\mathrm{e}^{\lambda t}\end{aligned}$$

where $\lambda_0 = \inf_{u\in V}(|\nabla u(x)|^2/|u(x)|^2)$. Picking $\lambda > 0$ large enough, using lemma 2.1 and the conditions of the theorem 2.3, we easily find that there exists a random variable $C(\omega)$ such that

$$Y_t^* \leqslant C(\omega)\cdot\sqrt{\log t} \qquad \text{a.s.}$$

Now the proof is complete. □

3. Examples

In this section we shall give several examples to illustrate our results derived above.

Example 3.1. Let us first consider the following stochastic Burgers-type equation. Assume $\nu > 0$, $\lambda > 0$, $u_0(x) \in R$ and for $t \geqslant 0$, $x \in (0, 1)$,

$$\begin{aligned} &\mathrm{d}u(t,x) = \left(\nu\frac{\partial^2 u(t,x)}{\partial^2 x} + \frac{1}{2}\frac{\partial}{\partial x}u^2(t,x)\right)\mathrm{d}t + (2t^3+5t)\,\mathrm{e}^{-2\lambda t}\,\mathrm{d}B_t(x) \\ &u(t,0) = u(t,1) = 0 \qquad t > 0 \\ &u(0,x) = u_0(x) \in V \qquad x \in [0,1] \end{aligned} \tag{3.1}$$

where $B_t(x)$ is an H-valued Wiener process with a bounded, continuous covariance function $q(x, y)$ with $\int_0^1 |q(x,x)|\,\mathrm{d}x \leqslant 1$ such that for $v(x) \in H$

$$(Qv)(x) = \int_0^1 q(x,y)v(y)\,\mathrm{d}y.$$

In order to apply theorem 2.1, we note that for any $\delta > 0$

$$\lim_{t\to\infty}\frac{2t^3+5t}{\mathrm{e}^{\delta t}} = 0.$$

Therefore, applying theorem 2.1, we conclude that equation (3.1) is exponentially stable almost certainly. Moreover, we have almost certainly

$$\limsup_{t\to\infty}\frac{\log|u(t)|}{t} \leqslant -\nu\lambda_0 \qquad \text{a.s.}$$

where $\lambda_0 = \inf_{y\in V}(|\nabla y(x)|^2/|y(x)|^2)$.

Example 3.2. Consider the following stochastic Burgers-type equation. Assume $\nu > 0$, $\alpha > 0$, $u_0 \in R$ and for $t \geqslant 0$, $x \in (0, 1)$,

$$\begin{aligned} &\mathrm{d}u(t,x) = \left(\nu\frac{\partial^2 u(t,x)}{\partial^2 x} + \frac{1}{2}\frac{\partial}{\partial x}u^2(t,x)\right)\mathrm{d}t + \alpha u(t,x)\,\mathrm{d}B_t(x) \\ &u(t,0) = u(t,1) = 0 \qquad t > 0 \\ &u(0,x) = u_0(x) \in V \qquad x \in [0,1] \end{aligned} \tag{3.2}$$

where $\lambda_0 = \inf_{y\in V}(|\nabla y(x)|^2/|y(x)|^2)$ and $B_t(x)$ is an H-valued Wiener process with a bounded continuous covariance $q(x, y)$, that is, there exists constant $M > 0$ such that $\sup_{x\in[0,1]}|q(x,x)| = M < \infty$.

Hence, by virtue of theorem 2.2, we have the fact that, if $2\nu\lambda_0 > \alpha^2 M$, equation (3.2) is exponentially stable almost certainly. Moreover, we have almost certainly

$$\limsup_{t\to\infty}\frac{\log|u(t)|}{t} \leqslant -\left(\frac{2\nu\lambda_0 - \alpha^2 M}{4}\right) \qquad \text{a.s.}$$

As is well known, under some conditions, such as stochastic bounded perturbation, the solution of stochastic equation may tend to infinity almost certainly. The following example establishes upper growth bounds for a class of stochastic Burgers equations.

Example 3.3. Consider the following stochastic Burgers equation. Assume $\nu > 0$, $u_0 \in R$ and for $t \geqslant 0$, $x \in (0, 1)$,

$$\begin{aligned}
&\mathrm{d}u(t,x) = \left(\frac{\partial^2 u(t,x)}{\partial^2 x} + \frac{\partial}{\partial x}u^2(t,x)\right)\mathrm{d}t + 2\,\mathrm{d}B_t(x) \\
&u(t,0) = u(t,1) = 0 \qquad t > 0 \\
&u(0,x) = u_0 x \in V \qquad x \in [0,1]
\end{aligned} \tag{3.3}$$

where $B_t(x)$ is an H-valued Wiener process with a bounded continuous covariance $q(x, y)$, that is, that there exists constant $M > 0$ such that $|q(x, x)| \leqslant M$.

Since, at the moment,

$$g(t,u) = 2$$

we have, by virtue of theorem 2.3, that there exists a positive random variable $C(\omega)$ such that

$$u^*(t,\omega) \leqslant C(\omega)\sqrt{\log t}$$

where $u^*(t, \omega) = \sup_{0\leqslant s\leqslant t} |u(s, \omega)|$.

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